

Slit maps in the study of equal-strength cavities in n -connected elastic planar domains

Y.A. ANTIPOV

Department of Mathematics, Louisiana State University
Baton Rouge LA 70803, USA

Abstract

The inverse problem of plane elasticity on n equal-strength cavities in a plane subjected to constant loading at infinity and in the cavities boundary is analyzed. By reducing the governing boundary value problem to the Riemann-Hilbert problem on a symmetric Riemann surface of genus $n - 1$ a family of conformal mappings from a parametric slit domain onto the n -connected elastic domain is constructed. The conformal mappings are presented in terms of hyperelliptic integrals and the zeros of the first derivative of the mappings are analyzed. It is shown that for any $n \geq 1$ there always exists a set of the loading parameters for which these zeros generate inadmissible poles of the solution.

1 Introduction

Analysis of the stresses induced by the presence of inclusions and cavities in an elastic matrix subjected to loading have been of interest for over a century [11], [14], [10]. Inverse problems of elasticity which concern problems of determination of the shapes of curvilinear inclusions and cavities with prescribed properties excite particular attention due to their relevance to the material design [3], [12], [4], [18], [2], [16]. A considerable amount of work examines equal-strain inclusions subjected to uniform loading with uniform distribution of stresses inside. By analyzing the stress distribution in composites with elliptic and ellipsoidal inclusions in two- and three-dimensional unbounded elastic bodies Eshelby [6] established that the stress fields are uniform in the interior of the inclusions provided the matrix is loaded uniformly at infinity. He also conjectured that there do not exist other shapes of inclusions with such a property. This conjecture was proved in the plane and anti-plane cases in [15]. An alternative proof for the antiplane case by the method of conformal mappings was proposed in [13].

Motivated by the problem of designing perforated structures of minimum weight Cherepanov [4] studied the inverse problem of elasticity for a plane uniformly loaded at infinity and having n holes. The boundary of the holes is subjected to constant normal and tangential traction, and the holes profile L_j ($j = 0, 1, \dots, n - 1$) are determined from an extra boundary condition. It states that the tangential normal stress σ_t is the same constant, σ , in all the contours L_j . For the solution, a conformal map of an n -connected slit domain \mathcal{D}^e into the elastic domain D^e , the exterior of the n holes, is applied. The map transforms the boundary value problem into two Schwarz problems of the theory of analytic functions on the n slits. The feature of the map $z = \omega(\zeta)$ employed in [4] is that it maps the point $\zeta = \infty$ into the point $z = \infty$, and the exterior of n parallel slits of the parametric plane into the n -connected domain D^e . In general, for such a map, unless $n = 1$ or $n = 2$, these slits do not lie in the same line. Cherepanov [4]

solved by quadratures the problem in the simply and doubly connected symmetric cases and also analyzed the periodic and doubly periodic problems. Vigdergauz [18] noticed and corrected an error in the computations [4] implemented for the symmetric doubly connected case when the slit domain is the exterior of the cuts $[-2, -1]$, $[1, 2]$, and the map $z = \omega(\zeta)$ has the property $\omega : \infty \rightarrow \infty$. For the symmetric case [4], the loading parameters a and b are real, and the two equal-strength holes exist if $|a/b| > 1$ [18].

To solve the Cherepanov problem for any n -connected domain, Vigdergauz [18] proposed to employ a circular map from the exterior of n -circles onto the n -connected elastic domain. Application of the Sherman integral representation reduced the resulting boundary value problem to integral equations solved numerically by the method of least squares. This method was further developed in [19] for doubly periodic structures by using integral representation with quasi-automorphic analogues of the Cauchy kernel and the numerical method of least squares for solving the governing integral equations. An alternative explicit representation in terms of the Weierstrass elliptic function for the profile of an inclusion in the case of a doubly periodic structure was given in [7]. The antiplane shear problem for two equal-strain inclusions by means of the Weierstrass zeta function was treated in [8].

The main goal of this paper is to derive a closed-form representation of a family of conformal mappings $z = \omega(\zeta)$, $\omega : \mathcal{D}^e \rightarrow D^e$ for $n = 2$ in the non symmetric case and for $n \geq 3$ and therefore to determine a family of the profiles of n equal-strength holes. In addition, we aim to study the poles of one of the Kolosov-Muskhelishvili complex potentials due to possible zeros of the derivative $\omega'(\zeta)$ of the conformal mapping. The stress field $\{\sigma_1, \sigma_2, \tau_{12}\}$ is expressible through two functions Φ and Ψ [11] which have to be analytic everywhere in the domain D^e . One of them, $\Phi(z)$, is a constant, while the second one has the form [4], p.918

$$\Psi(\omega(\zeta)) = \frac{F_+(\zeta) + F_-(\zeta)}{2\omega'(\zeta)}, \quad \zeta \in \mathcal{D}^e = \mathbb{C} \setminus l, \quad (1.1)$$

where $F_{\pm}(\zeta)$, the solutions to certain Schwarz problems, are analytic functions in $\mathbb{C} \setminus l$, and l is the union of the slits l_j , $j = 0, 1, \dots, n-1$, in the parametric ζ -plane. If $\omega'(\zeta)$ has zeros in the slit domain \mathcal{D}^e , then the potential $\Psi(z)$ has inadmissible poles at the images of these zeros. Note that the impact of the possible zeros on the solution was overlooked not only in [4], but also in [18]. Indeed, the function $\Psi(\zeta)$ has to be analytic in the exterior of n circles, $\tilde{\mathcal{D}}^e$. Since it is expressed through the function $\omega'(\zeta)$ and another analytic function as [18], p.519

$$\Psi(\zeta) = \frac{aF'(\zeta)}{\omega'(\zeta)}, \quad (1.2)$$

the zeros of $\omega'(\zeta)$, if exist, generate inadmissible poles of the function $\Psi(\zeta)$.

In Section 2, we formulate the problem and follow [4] to reduce it to two Schwarz problems. The case $n = 1$ is analyzed in Section 3. It is shown, that the solution is an ellipse, and when $\gamma = |a/b| > 1$, the function $\Psi(\zeta)$ is free of poles. Otherwise, if $\gamma \leq 1$, it has either two inadmissible poles or is singular on the contour. Here,

$$a = \frac{1}{2}(\sigma - p) + i\tau, \quad b = \frac{1}{2}(\sigma_2^\infty - \sigma_1^\infty) + i\tau^\infty, \quad (1.3)$$

$\sigma_1^\infty, \sigma_2^\infty, \tau^\infty$ are constant stresses applied at infinity, p and τ are the traction components applied to the holes boundaries, and $\sigma = \sigma_1^\infty + \sigma_2^\infty - p$.

Section 4 gives an integral representation in terms of elliptic integrals of the mapping function for $n = 2$ in the general not necessarily symmetric case. The map has two free parameters and, in addition, has two free scaling parameters. It is shown that when $\gamma = |a/b| > 1$, then the solution always exists. Otherwise, if $\gamma < 1$, then the function $\Psi(\zeta)$ has four poles, while the contours L_1 and L_2 , the profiles of the holes, may (for sufficiently small values of the parameter γ) or may not intersect. If $\gamma \rightarrow 1^\pm$, then the contours intend to become segments, and the function $\Psi(z)$ is singular on the contours in the limiting case $\gamma = 1$.

In Section 5, we analyze the triple connected case. To pursue the goal to describe the general family of equal-strength cavities, we construct the most general form of the conformal map possible in the case $n = 3$. It maps the exterior of three slits $[-1/k, -1]$, $[k_1, k_2]$, and $[1, 1/k]$ ($0 < k < 1$, $-1 < k_1 < k_2 < 1$) into the triply connected domain D^e , while the infinite point $z = \infty$ is the image of a point $\zeta_\infty = \zeta'_\infty + i\zeta''_\infty$. Thus, in addition to two scaling parameters it has five real free parameters, k , k_1 , k_2 , ζ'_∞ , and ζ''_∞ . The governing boundary value problem reduces to a symmetric Riemann-Hilbert problem on a genus-2 Riemann surface and similarly to [1] is solved exactly. It is shown that if ζ_∞ is a finite point, then regardless of the values of the parameter γ the function $\Psi(z)$ has always inadmissible poles in D^e , and the solution does not exist.

In Section 6, we identify a family of n ($n \geq 3$) equal-strength cavities by constructing a conformal map which transforms a slit domain into an n -connected domain D^e . We choose the parametric domain \mathcal{D}^e as the union of slits in the real axis and assume that $\omega : \infty \rightarrow \infty$. The conformal map with such properties has $2n - 2$ free real parameters. In particular, for the case $n = 3$ and $\zeta_\infty = \infty$, we show that the family of conformal mappings is four-parametric (the slits are $[-1, k_1]$, $[k_2, k_3]$, and $[k_4, 1]$) and if $\gamma > 1$, then the solution is free of poles. Otherwise, if $\gamma < 1$, it has six poles, and the solution does not exist.

2 Formulation

Consider the following problem of plane elasticity [4]:

Let an infinite isotropic plane subjected to constant stresses at infinity, $\sigma_1 = \sigma_1^\infty$, $\sigma_2 = \sigma_2^\infty$, and $\tau_{12} = \tau^\infty$, have n holes D_0, D_1, \dots, D_{n-1} . Assume that constant normal and tangential traction components are applied to their boundaries L_j , $\sigma_n = p$, $\tau_{nt} = \tau$, $j = 0, 1, \dots, n-1$. Find the shape and location of the holes such that the tangent normal stress σ_t is constant, $\sigma_t = \sigma$, in all the contours L_j .

Let $\Phi(z)$ and $\Psi(z)$ ($z = x_1 + ix_2$) be the Kolosov-Muskhelishvili potentials of the problem. These functions are analytic everywhere in the n -connected domain $D^e = \mathbb{C} \setminus D$, $D = \cup_{j=0}^{n-1} D_j$ and continuous in D^e up to its boundary. The equilibrium equations of plane elasticity are satisfied if the stresses are [11]

$$\sigma_1 + \sigma_2 = 4 \operatorname{Re} \Phi(z), \quad \sigma_2 - \sigma_1 + 2i\tau_{12} = 2[\bar{z}\Phi'(z) + \Psi(z)]. \quad (2.1)$$

The stresses and the traction vector components on the boundaries are connected by the relations

$$\sigma_t + \sigma_n = \sigma_1 + \sigma_2, \quad \sigma_t - \sigma_n + 2i\tau_{nt} = e^{2i\alpha(z)}(\sigma_2 - \sigma_1 + 2i\tau_{12}). \quad (2.2)$$

Here, $\alpha(z)$ is the angle between the positive direction of the x_1 -axis and the external normal n to the cavity boundary. At infinity, $\Phi(z) = \frac{1}{4}(\sigma_1^\infty + \sigma_2^\infty) + O(z^{-2})$. This implies

[4] $\Phi(z) = \frac{1}{4}(\sigma_1^\infty + \sigma_2^\infty)$ everywhere in D^e and $\sigma = \sigma_1^\infty + \sigma_2^\infty - p$. As for the analytic function $\Psi(z)$, it has to satisfy the conditions

$$\begin{aligned}\Psi(z) &= ae^{-2ia(z)}, \quad z \in L = \bigcup_{j=0}^{n-1} L_j, \quad a = \frac{1}{2}(\sigma - p) + i\tau, \\ \Psi(z) &= b + O(z^{-2}), \quad z \rightarrow \infty, \quad b = \frac{1}{2}(\sigma_2^\infty - \sigma_1^\infty) + i\tau^\infty.\end{aligned}\tag{2.3}$$

It is known [5], [9] that there exists an analytic function $z = \omega(\zeta)$ that conformally maps the extended complex ζ -plane $\mathbb{C} \cup \infty$ cut along n segments parallel to the real ζ -axis onto the n -connected domain D^e in the z -plane. Such a function is a one-to-one map. The infinite point $z = \infty$ is the image of a certain point $\zeta = \zeta_\infty$, and in the vicinity of that point the conformal map $\omega(\zeta)$ can be represented as

$$\omega(\zeta) = \frac{c_{-1}}{\zeta - \zeta_\infty} + c_0 + \sum_{j=1}^{\infty} c_j(\zeta - \zeta_\infty)^j.\tag{2.4}$$

The boundary condition in (2.3) written in the ζ -plane read [4]

$$\psi(\zeta)\omega'(\zeta) + \overline{a\omega'(\zeta)} = 0, \quad \zeta \in l = \bigcup_{j=0}^{n-1} l_j,\tag{2.5}$$

where $\psi(\zeta) = \Psi(z)$. The problem is significantly simplified for the two analytic functions $F_\pm(\zeta) = [\psi(\zeta) \pm \bar{a}]\omega'(\zeta)$. The new functions satisfy the following two Schwarz boundary conditions on the cuts l :

$$\operatorname{Re} F_+(\zeta) = 0, \quad \operatorname{Im} F_-(\zeta) = 0, \quad \zeta \in l.\tag{2.6}$$

If these two functions are known, then the functions $\omega'(\zeta)$ and $\psi(\zeta)$ are determined by

$$\omega'(\zeta) = \frac{F_+(\zeta) - F_-(\zeta)}{2\bar{a}}, \quad \psi(\zeta) = \frac{F_+(\zeta) + F_-(\zeta)}{2\omega'(\zeta)}, \quad \zeta \in \mathbb{C} \setminus l.\tag{2.7}$$

3 One hole

If $n = 1$, then with no loss of generality l_0 and the point ζ_∞ may be selected as $[-1, 1]$ and ∞ , respectively. Consequently the representation (2.4) in a neighborhood of the infinite point reads

$$\omega(\zeta) = c_{-1}\zeta + c_0 + \sum_{j=1}^{\infty} \frac{c_j}{\zeta^j}.\tag{3.1}$$

Such a map is defined up to one real parameter, and we assume that $\operatorname{Im} c_{-1} = 0$. This and (2.3) imply that at infinity the function $\omega'(\zeta)$ is bounded, $\omega'(\zeta) \sim c_{-1}$, while the function $\psi(\zeta)$ decays as $\psi(z) = b + O(\zeta^{-2})$.

Let \mathcal{R} be the two-sheeted genus-0 Riemann surface of the algebraic function $u^2 = p_1(\zeta)$, where $p_1(\zeta) = \zeta^2 - 1$. Fix the single branch $f(\zeta)$ of $p_1^{1/2}(\zeta)$ in the ζ -plane cut along l_0 by the condition $p_1^{1/2}(\zeta) \sim z$, $\zeta \rightarrow \infty$. This branch is pure imaginary on the sides of the cut and real in the real axis outside the cut. Define next

$$u = \begin{cases} \sqrt{p_1(\zeta)}, & \zeta \in \mathbb{C}_1, \\ -\sqrt{p_1(\zeta)}, & \zeta \in \mathbb{C}_2, \end{cases}\tag{3.2}$$

where \mathbb{C}_1 and \mathbb{C}_2 are the two sheets of the surface \mathcal{R} and $f(\zeta) = \sqrt{p_1(\zeta)}$. The Schwarz problems (2.6) are equivalent to the following simple Riemann-Hilbert problems on the surface \mathcal{R} :

$$\begin{aligned} [iF_+(\zeta)]^+ - [iF_+(\zeta)]^- &= 0, \quad [F_-(\zeta)]^+ - [F_-(\zeta)]^- = 0, \quad \zeta \in l_0, \\ F_\pm(\zeta) &= c_{-1}(b \pm \bar{a}) + O(\zeta^{-2}), \quad \zeta \rightarrow \infty, \end{aligned} \quad (3.3)$$

and their solutions are rational functions on the surface \mathcal{R} . These functions have simple poles (in the sense of the theory of Riemann surfaces [17]) at the branch points of the surface \mathcal{R} . We have

$$F_+(\zeta) = \frac{A_1^+ + A_2^+ \zeta}{f(\zeta)} + iA_0^+, \quad F_-(\zeta) = \frac{i(A_1^- + A_2^- \zeta)}{f(\zeta)} + A_0^-, \quad (3.4)$$

where A_j^\pm ($j = 0, 1, 2$) are arbitrary real constants. Due to the asymptotics (3.3) of the functions $F_\pm(\zeta)$ we have from (3.4)

$$\begin{aligned} A_1^\pm &= 0, \quad A_2^+ = c_{-1} \operatorname{Re}(b + \bar{a}), \quad A_2^- = c_{-1} \operatorname{Im}(b - \bar{a}), \\ A_0^+ &= c_{-1} \operatorname{Im}(b + \bar{a}), \quad A_0^- = c_{-1} \operatorname{Re}(b - \bar{a}). \end{aligned} \quad (3.5)$$

Substituting these coefficients into (3.4) and then into (2.7) we derive

$$\omega'(\zeta) = \frac{c_{-1}}{2} \left[m_- + \frac{m_+ \zeta}{f(\zeta)} \right], \quad m_\pm = 1 \pm \frac{\bar{b}}{a}, \quad \psi(\zeta) = \bar{a} \frac{(a+b)\zeta - (a-b)f(\zeta)}{(\bar{a} + \bar{b})\zeta + (\bar{a} - \bar{b})f(\zeta)}. \quad (3.6)$$

Notice that

$$\int_{l_0} \omega'(\zeta) d\zeta = 0, \quad (3.7)$$

and the map is one-to-one. The conformal map $z = \omega(\zeta)$ is defined up to an additive constant B , has the form [4]

$$\omega(\zeta) = \frac{c_{-1}}{2} [m_- \zeta + m_+ (\zeta^2 - 1)^{1/2}] + B, \quad (3.8)$$

and $z = \omega(\zeta)$, $\zeta \in l_0$, is a parametric equation of a family of ellipses (c_{-1} is an arbitrary nonzero real parameter).

Now, the function $\psi(\zeta)$ has to be analytic everywhere in $\mathbb{C} \setminus l_0$. Possible singularities of this function coincide with the zeros of the derivative of the map or, equivalently, with the zeros of the function $\eta(\zeta) = m_- \sqrt{\zeta^2 - 1} + m_+ \zeta$. To determine their number, consider a simple closed positively oriented contour $\Gamma = \Gamma_R \cup [R - i0, -1 - i0] \cup [-1 + i0, R + i0]$, where $R > 1$ and $\Gamma_R = \{|\zeta| = R\}$. The number of zeros of $\eta(\zeta)$ inside the contour Γ is given by

$$Z = \frac{1}{2\pi i} \int_{\Gamma} \frac{\eta'(\zeta) d\zeta}{\eta(\zeta)}. \quad (3.9)$$

Computing the integral over the contour Γ_R , letting $R \rightarrow \infty$, and transforming the integral over the contour l_0 we obtain

$$Z = 1 - \frac{m_+ m_-}{\pi(m_+^2 - m_-^2)} I, \quad (3.10)$$

where

$$I = \int_{-1}^1 \frac{d\xi}{\sqrt{1 - \xi^2}(\xi^2 + \mu^2)}, \quad \mu^2 = \frac{m_-^2}{m_+^2 - m_-^2}. \quad (3.11)$$

This integral can be computed by making the subsequent substitutions $\xi = \cos \phi$ and $e^{i\theta} = w$ and applying the theory of residues in the w -plane,

$$I = \pm \frac{\pi(m_+^2 - m_-^2)}{m_+ m_-}, \quad \left| \frac{a}{b} \right|^{\pm 1} > 1. \quad (3.12)$$

Substituting this into formula (3.10) we derive that $Z = 0$ if $|a/b| > 1$ and $Z = 2$ if $|a/b| < 1$. These poles can be easily determined

$$\zeta_{1,2} = \pm \frac{i}{2} \left(\sqrt{\frac{\bar{b}}{a}} - \sqrt{\frac{\bar{a}}{b}} \right). \quad (3.13)$$

In the case $a = \bar{b}$, the function $\eta'(\zeta)/\eta(\zeta)$ has two distinct singularities which fall into the contour l_0 , $\zeta_{1,2} = \pm \sin(\arg a)$, and if $a = b$, then the singular point is $\zeta = 0$. This proves that the functions $\psi(\zeta)$ and $\Psi(z)$ are analytic in the exteriors of the contours l_0 and L_0 , respectively, only if the parameters of the problem satisfy the condition $|a| > |b|$. Otherwise the function $\Psi(z)$ is either not analytic in the exterior of the contour L_0 , or not continuous in D^e up to its boundary. If $|a| < |b|$, then the function $\Psi(z)$ has two poles, and if $|a| = |b|$, then it is singular on the contour L_0 . Thus, if $|a| \leq |b|$, then the problem does not have solutions.

4 Two holes

Every doubly connected domain D^e may be conformally mapped by a function $\zeta = \omega^{-1}(z)$ onto a slit domain \mathcal{D}^e , the extended complex ζ -plane cut along the two cuts $l_0 = [-1/k, -1]$ and $l_1 = [1, 1/k]$, where $k \in (0, 1)$. Moreover, it is possible to choose the map such that the infinite point $z = \infty$ falls into a point ζ_∞ in the open segment $(-1, 1)$ of the real ζ -axis. Such a map can be considered as the composition of a conformal map of the domain D^e onto an annulus $\{1 < |\zeta| < r\}$, the transformation of rotation which puts the image of the point $z = \infty$ into the segment $(1, r)$ of the real axis, and the mapping of the annulus onto the slit domain \mathcal{D}^e by means of an elliptic integral.

Since $\zeta = \infty$ is the ω -image of a regular point of the domain D^e , the function $\psi(\zeta)$ is bounded at infinity and also

$$\omega(\zeta) = d_0 + \frac{d_1}{\zeta} + \frac{d_2}{\zeta^2} + \dots, \quad \omega'(\zeta) = -\frac{d_1}{\zeta^2} - \frac{2d_2}{\zeta^3} - \dots, \quad \zeta \rightarrow \infty. \quad (4.1)$$

In a neighborhood of the point $\zeta_\infty \in (-1, 1)$ the function $\omega(\zeta)$ admits the expansion (2.4) and therefore

$$\psi(\zeta) = b + O((\zeta - \zeta_\infty)^2), \quad \omega'(\zeta) = -\frac{c_{-1}}{(\zeta - \zeta_\infty)^2} + c_1 + 2c_2(\zeta - \zeta_\infty) + \dots, \quad \zeta \rightarrow \infty. \quad (4.2)$$

Thus the two Schwarz problems (2.6) need to be solved in the class of functions which meet the conditions

$$F_\pm(\zeta) = O\left(\frac{1}{\zeta^2}\right), \quad \zeta \rightarrow \infty, \\ F_\pm(\zeta) = -\frac{c_{-1}(b \pm \bar{a})}{(\zeta - \zeta_\infty)^2} + O(1), \quad \zeta \rightarrow \zeta_\infty. \quad (4.3)$$

Similarly to the previous case we introduce the elliptic surface \mathcal{R} of the algebraic function $u^2 = p_2(\zeta)$, where $p_2(\zeta) = (\zeta^2 - 1)(\zeta^2 - 1/k^2)$. A single branch $f(\zeta)$ of the

function $p_2^{1/2}(\zeta)$ is fixed in the ζ -plane cut along the two segments l_0 and l_1 by the condition $p_2^{1/2}(\zeta) \sim \zeta^2$, $\zeta \rightarrow \infty$. This branch is pure imaginary on the sides of the cuts, $f^\pm(\zeta) = \mp i(-1)^j \sqrt{|p_2(\zeta)|}$, $\zeta \in l_j$, $j = 0, 1$, and real for $\zeta = \xi$ lying outside the cuts in the real axis, $f(\xi) > 0$, $|\xi| > 1/k$, and $f(\xi) < 0$, $-1 < \xi < 1$.

The Schwarz problems (2.6), (4.3) can be rewritten as the two Riemann-Hilbert problems on the elliptic surface \mathcal{R}

$$[iF_+(\zeta)]^+ - [iF_+(\zeta)]^- = 0, \quad [F_-(\zeta)]^+ - [F_-(\zeta)]^- = 0, \quad \zeta \in l = l_0 \cup l_1, \quad (4.4)$$

These functions have simple poles at the branch point of the surface and due to (4.3) have order-2 poles with zero residues at the point ζ_∞ in both sheets. The infinite points of the surface are order-2 zeros of the functions $F_+(\zeta)$ and $F_-(\zeta)$. The solutions of these problems are the following rational function in the surface \mathcal{R} :

$$\begin{aligned} F_-(\zeta) &= \frac{1}{(\zeta - \zeta_\infty)^2} \left(A_0^- + \frac{i(A_1^- + A_2^- \zeta + A_3^- \zeta^2)}{f(\zeta)} \right), \\ F_+(\zeta) &= \frac{1}{(\zeta - \zeta_\infty)^2} \left(iA_0^+ + \frac{A_1^+ + A_2^+ \zeta + A_3^+ \zeta^2}{f(\zeta)} \right), \end{aligned} \quad (4.5)$$

where A_j^\pm are real constants to be determined.

Denote

$$c_{-1} = c' + ic'', \quad b \pm \bar{a} = \alpha^\pm + i\beta^\pm. \quad (4.6)$$

Due to (2.3) the four parameters α^\pm and β^\pm are expressed through the loading data as

$$\alpha^+ = \sigma_2^\infty - p, \quad \alpha^- = p - \sigma_1^\infty, \quad \beta^+ = \tau^\infty - \tau, \quad \beta^- = \tau^\infty + \tau. \quad (4.7)$$

On expanding the functions (4.5) in a neighborhood of the point $\zeta = \zeta_\infty$ and satisfying the second condition in (4.3) we determine A_0^\pm

$$A_0^+ = -c'\beta^+ - c''\alpha^+, \quad A_0^- = -c'\alpha^- + c''\beta^-, \quad (4.8)$$

and derive four real equations for the other six coefficients

$$\begin{aligned} A_1^\pm + \zeta_\infty A_2^\pm + \zeta_\infty^2 A_3^\pm &= d^\pm, \\ A_2^\pm + 2\zeta_\infty A_3^\pm &= \frac{d^\pm p_2'(\zeta_\infty)}{2p_2(\zeta_\infty)}, \end{aligned} \quad (4.9)$$

where

$$d^+ = \sqrt{|p_2(\zeta_\infty)|}(c'\alpha^+ - c''\beta^+), \quad d^- = \sqrt{|p_2(\zeta_\infty)|}(c'\beta^- + c''\alpha^-). \quad (4.10)$$

Denoting

$$A_0 = iA_0^+ - A_0^-, \quad A_j = A_j^+ - iA_j^-, \quad j = 1, 2, 3, \quad (4.11)$$

and implying formula (2.7) we obtain the derivative of the conformal map

$$\omega'(\zeta) = \frac{1}{2\bar{a}(\zeta - \zeta_\infty)^2} \left(A_0 + \frac{A_1 + A_2\zeta + A_3\zeta^2}{f(\zeta)} \right). \quad (4.12)$$

In general, the map $z = \omega(\zeta)$ given by (4.12) is a multi-valued function. It is a one-to-one map if

$$\int_{l_0} \omega'(\zeta) d\zeta = 0, \quad \int_{l_1} \omega'(\zeta) d\zeta = 0. \quad (4.13)$$

The two integrals over the loops l_0 and l_1 vanish if the coefficients A_j^\pm ($j = 1, 2, 3$) solve the four equations

$$\begin{aligned} A_1^\pm I_0^- + A_2^\pm I_1^- + A_3^\pm I_2^- &= 0, \\ A_1^\pm I_0^+ - A_2^\pm I_1^+ + A_3^\pm I_2^+ &= 0, \end{aligned} \quad (4.14)$$

where

$$I_j^\pm = \int_1^{1/k} \frac{\xi^j d\xi}{(\xi \pm \zeta_\infty)^2 \sqrt{|p_2(\xi)|}}, \quad j = 0, 1, 2. \quad (4.15)$$

The system of eight equations (4.9), (4.14) for the six unknowns A_j^\pm ($j = 1, 2, 3$) has rank 6: the third and fourth equations in (4.14) are identically satisfied provided A_j^\pm solve equations (4.9) and the first and second equations in (4.14). Upon solving the system we express the coefficients A_j^\pm through the four problem parameters α^\pm and β^\pm and the four conformal map parameters c' , c'' , ζ_∞ , and k in the form

$$\begin{aligned} A_1^\pm &= \frac{d^\pm}{\lambda_0} [\zeta_\infty (\lambda_1 \zeta_\infty - 2) I_1^- + (1 - \lambda_1 \zeta_\infty) I_2^-], \\ A_2^\pm &= \frac{d^\pm}{\lambda_0} [\zeta_\infty (2 - \lambda_1 \zeta_\infty) I_0^- + \lambda_1 I_2^-], \\ A_3^\pm &= -\frac{d^\pm}{\lambda_0} [(1 - \lambda_1 \zeta_\infty) I_0^- + \lambda_1 I_1^-]. \end{aligned} \quad (4.16)$$

Here,

$$\lambda_0 = \zeta_\infty^2 I_0^- - 2\zeta_\infty I_1^- + I_2^-, \quad \lambda_1 = \frac{p_2'(\zeta_\infty)}{2p_2(\zeta_\infty)}. \quad (4.17)$$

The map itself, in addition to the four real parameters c' , c'' , ζ_∞ , and k , has an additive constant, B ,

$$\omega(\zeta) = \frac{1}{2a} \left[-\frac{A_0}{\zeta - \zeta_\infty} + \int_{\zeta_0}^{\zeta} \frac{(A_1 + A_2 \xi + A_3 \xi^2) d\xi}{(\xi - \zeta_\infty)^2 \sqrt{p_2(\xi)}} \right] + B, \quad (4.18)$$

where the path of integration $\zeta_0 \zeta$ does not pass through the point $\zeta_\infty \in (-1, 1)$. When a point ζ traverses the contours l_0 or l_1 , the point $z = \omega(\zeta)$ traverses the contours L_0 or L_1 , respectively.

As in the case $n = 1$, the function $\psi(\zeta)$ may have inadmissible poles in the exterior of l . They coincide with the zeros of the derivative $\omega'(\zeta)$ or, equivalently, with the zeros of the function

$$\eta(\zeta) = A_1 + A_2 \zeta + A_3 \zeta^2 + A_0 p_2^{1/2}(\zeta). \quad (4.19)$$

The number of inadmissible poles of the function $\psi(\zeta)$ is determined by

$$Z = \frac{1}{2\pi i} \left(\int_{l_0} + \int_{l_1} + \lim_{R \rightarrow \infty} \int_{\Gamma_R} \right) \frac{\eta'(\zeta) d\zeta}{\eta(\zeta)}, \quad (4.20)$$

where the positive direction is chosen such that the interior of circle $\Gamma_R = \{|\zeta| = R\}$ and the exterior of the cuts l is on the left. Noticing that

$$\eta'(\zeta) = A_2 + 2A_3 \zeta + \frac{A_0 \zeta (2\zeta^2 - 1 - 1/k^2)}{p_2^{1/2}(\zeta)} \sim 2(A_0 + A_3) \zeta, \quad \zeta \rightarrow \infty, \quad (4.21)$$

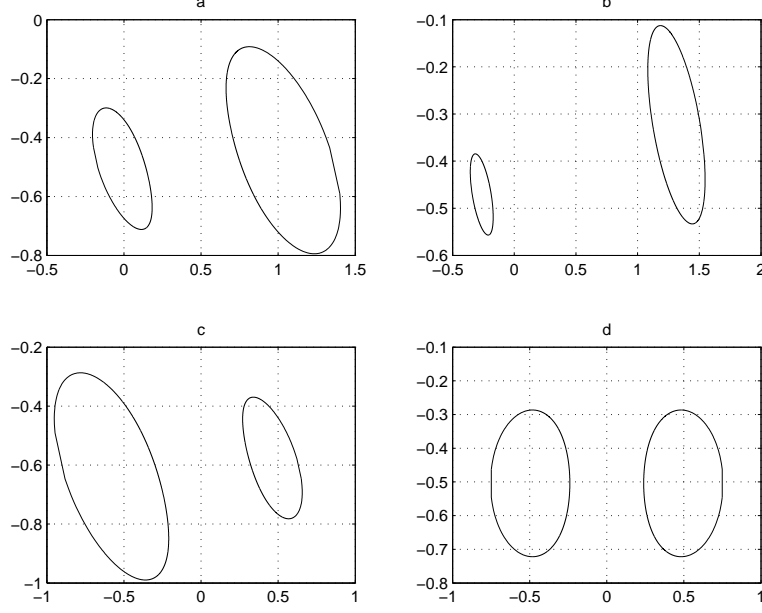


Figure 1: Two equal-strength holes when (a-c) $\alpha^+ = -3$, $\alpha^- = 4$, $\beta^+ = 0$, $\beta^- = 2$ ($\gamma = 3.2558$) for some values of the conformal mapping parameters. (a): $c' = 1$, $c'' = 0.1$, $k = 0.1$, $\zeta_\infty = 0.3$; (b): $c' = 1$, $c'' = 0.1$, $k = 0.5$, $\zeta_\infty = 0.3$; (c): $c' = 1$, $c'' = 0.1$, $k = 0.1$, $\zeta_\infty = -0.3$. The symmetric case when $\alpha^+ = -3$, $\alpha^- = 4$, $\beta^+ = \beta^- = 0$ ($\gamma = 7$) $c' = 1$, $c'' = 0$, $k = 0.1$, and $\zeta_\infty = 0$ is shown in (d).

we can transform formula (4.21) as

$$Z = 2 + \frac{1}{2\pi i} \left(\int_{-1/k}^{-1} - \int_1^{1/k} \right) \left(\frac{\eta_0^-(\xi)}{\eta^-(\xi)} - \frac{\eta_0^+(\xi)}{\eta^+(\xi)} \right) \frac{d\xi}{\sqrt{|p_2(\xi)|}}. \quad (4.22)$$

where

$$\begin{aligned} \eta^\pm(\xi) &= \pm i \sqrt{|p_2(\xi)|} A_0 + A_1 + A_2 \xi + A_3 \xi^2, \\ \eta_0^\pm(\xi) &= \mp i A_0 \xi (2\xi^2 - 1 - 1/k^2) + \sqrt{|p_2(\xi)|} (A_2 + 2A_3 \xi). \end{aligned} \quad (4.23)$$

On evaluating the integrals in (4.22) we conclude that as in the case of one hole the key parameter is $\gamma = |a/b|$. If $\gamma > 1$, then $Z = 0$, and the function $\psi(\zeta)$ is analytic in the exterior of the cuts $l = l_0 \cup l_1$ and continuous up to the boundary l . When $\gamma < 1$, the function $\psi(\zeta)$ has four poles, and in the case $\gamma = 1$ the function $\psi(\zeta)$ has singularities in the cuts sides. As before, when $|a| \leq |b|$ the function $\Psi(z)$ is either not analytic everywhere in the exterior of the holes, or is not continuous up to the boundary $L = L_1 \cup L_2$. Thus in the case $\gamma \leq 1$ the problem does not have solutions.

The profiles L_j of equal-strength holes are determined from formula (4.18) as

$$\begin{aligned} z &= \frac{1}{2\bar{a}} \left[-\frac{A_0}{\zeta - \zeta_\infty} + I_- \pm iJ(-1/k, \zeta) \right], \quad \zeta \in l_0^\pm, \quad z \in L_0, \\ z &= \frac{1}{2\bar{a}} \left[-\frac{A_0}{\zeta - \zeta_\infty} + I_+ \mp iJ(1, \zeta) \right], \quad \zeta \in l_1^\pm, \quad z \in L_1, \end{aligned} \quad (4.24)$$

where

$$I_\pm = \int_{\Gamma_\pm} \frac{(A_1 + A_2 \xi + A_3 \xi^2) d\xi}{(\xi - \zeta_\infty)^2 p_2^{1/2}(\xi)},$$

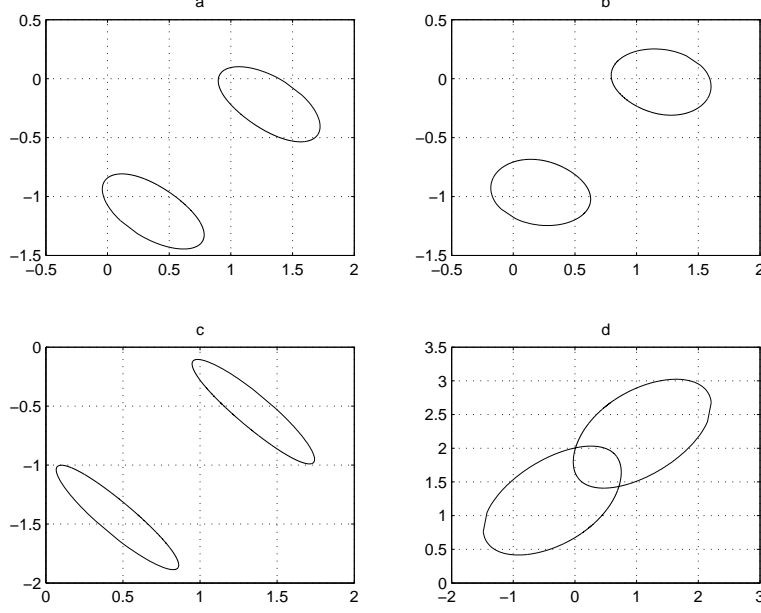


Figure 2: Two equal-strength holes when (a-d) $c' = c'' = 1$, $k = 0.1$, $\zeta_\infty = 0$, $\sigma_1^\infty = 1$, $\sigma_2^\infty = 2$ for some values of the other loading parameters. (a): $\tau^\infty = 1$, $\tau = 0$, $p = 5$ ($\alpha^+ = -3$, $\alpha^- = 4$, $\beta^+ = 1$, $\beta^- = 1$, $\gamma = 3.1305$); (b): $\tau^\infty = 1$, $\tau = -5$, $p = 5$ ($\gamma = 5.4589$); (c): $\tau^\infty = 1$, $\tau = 1$, $p = 3$ ($\gamma = 1.6125$); (d): $\tau^\infty = -5$, $\tau = 1$, $p = 3$ ($\gamma = 0.3588$). In case (d), $\gamma < 1$, the function $\Psi(z)$ has four inadmissible poles, and the solution does not exist.

$$J(d, \zeta) = \int_d^\zeta \frac{(A_1 + A_2\xi + A_3\xi^2)d\xi}{(\xi - \zeta_\infty)^2 \sqrt{|p_2(\xi)|}}. \quad (4.25)$$

Here, Γ_\pm are the segments with the starting and terminal points ζ_0 and ± 1 , respectively, and with no loss we assumed $B = 0$ and $\zeta_0 = -i$.

Some typical shapes of two equal-strength holes are shown in Figures 1 - 3. In Figures 1(a)-(c), $c' = 1$ and the loading parameters α^\pm and β^\pm do not change, while the conformal mapping parameters c'' , e_∞ , and k vary. Figure 1(d) relates to the symmetric case when the parameters a and b are real, $c'' = 0$ and $\zeta_\infty = 0$. In the cases 1(a)-(d) $\gamma > 1$, and the function $\Psi(z)$ does not have poles in the exterior of the holes. For Figures 2(a)-2(d), the conformal mapping parameters are kept unchanged and $\sigma_1^\infty = 1$, $\sigma_2^\infty = 2$. The other loading parameters, τ^∞ , τ , and p vary. In Figures 2(a)-(c), $\gamma > 1$, and the solution exists, while in Figure 2(d), $\gamma < 1$, and the solution has inadmissible poles. Referring to Figure 2(d), we observe that the contours L_0 and L_1 intersect each other, that is in the case $\gamma < 1$ the presence of inadmissible poles of the function $\Psi(z)$ may not be the only one feature which indicates that the solution does not exist. When γ approaches 1 and either $\gamma > 1$ or $\gamma < 1$, the contours become slim, and in the limit, when $\gamma = 1$, the contours L_0 and L_1 become segments, and the function $\Psi(z)$ is singular in these contours.

The solution may be significantly simplified in the symmetric case when $c'' = 0$, $\zeta_\infty = 0$, $\tau = \tau^\infty = 0$. Then

$$a = \frac{\sigma - p}{2}, \quad b = \frac{\sigma_2^\infty - \sigma_1^\infty}{2}, \quad \alpha^\pm = b \pm a, \quad \beta^\pm = 0,$$

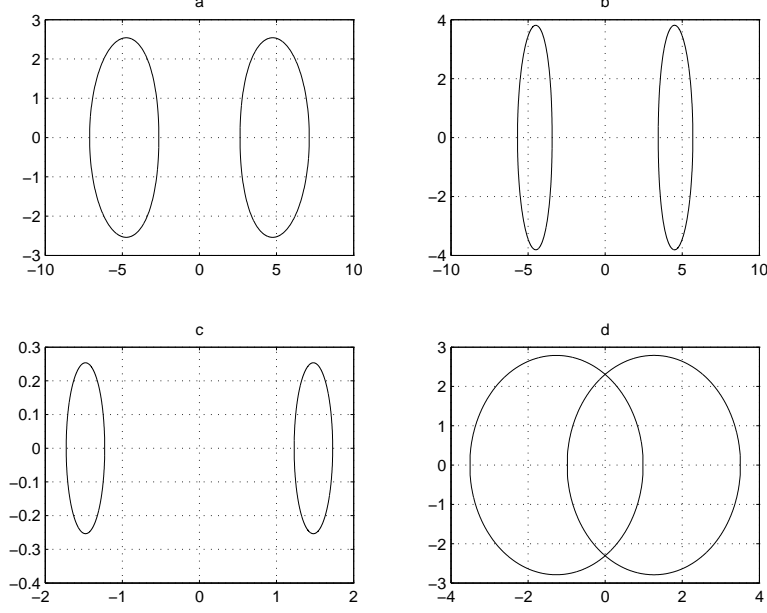


Figure 3: Two equal-strength symmetric holes for $\zeta_\infty = \infty$, $c' = 1$, $c'' = 0$, $\tau = \tau^\infty = 0$, $a = 1$. The values of the other parameters are (a) $k = 0.1$, $b = 0$, ($\gamma = \infty$), (b) $k = 0.1$, $b = 0.5$ ($\gamma = 2$), (c) $k = 0.5$, $b = 0$ ($\gamma = \infty$), (d) $k = 0.5$, $b = 10$ ($\gamma = 0.1$, and the function $\Psi(z)$ has four inadmissible poles).

$$\begin{aligned}
d^+ &= \frac{c'\alpha^+}{k}, \quad d^- = 0, \quad A_0^+ = 0, \quad A_0^- = -c'\alpha^-, \\
A_1^+ &= \frac{c'\alpha^+}{k}, \quad A_1^- = 0, \quad A_2^\pm = 0, \quad A_3^- = 0, \quad A_3^+ = -\frac{c'\alpha^+ I_0^-}{k I_2^-},
\end{aligned} \tag{4.26}$$

and the derivative of the conformal map has the form

$$\omega'(\zeta) = \frac{c'}{2a\zeta^2} \left[\alpha^- + \frac{\alpha^+(1 - \zeta^2 I_0^- / I_2^-)}{k p_2^{1/2}(\zeta)} \right]. \tag{4.27}$$

If it is assumed that as in [4] the function $z = \omega(\zeta)$ maps the infinite point $\zeta = \infty$ into the infinite point $z = \infty$, and the case is symmetric that is $c'' = 0$, $\tau = \tau^\infty = 0$, then

$$F_-(\zeta) = c'\alpha^-, \quad F_+(\zeta) = \frac{c'\alpha^+(\zeta^2 - I_2/I_0)}{p^{1/2}(\zeta)}, \tag{4.28}$$

and the function $\omega'(\zeta)$ can be represented as

$$\omega'(\zeta) = \frac{c'}{2a} \left[-\alpha^- + \frac{\alpha^+(\zeta^2 - I_2/I_0)}{p_2^{1/2}(\zeta)} \right]. \tag{4.29}$$

Here,

$$I_j = \int_1^{1/k} \frac{\xi^j d\xi}{\sqrt{|p_2(\xi)|}}, \quad j = 0, 2. \tag{4.30}$$

It is directly verified that the function (4.29) satisfy the conditions (4.13), and it is a one-to-one map. Possible poles of the function $\psi(\zeta)$ coincide with the zeros of the derivative $\omega'(\zeta)$ or, equivalently, with the zeros of the function

$$\eta(\zeta) = \alpha_1(\zeta^2 - \lambda) - \alpha_0 p_2^{1/2}(\zeta), \quad \alpha_0 = \alpha^-, \quad \alpha_1 = \alpha^+, \quad \lambda = I_2/I_0. \tag{4.31}$$

Due to the symmetry of the contours l_0 and l_1 the number of inadmissible poles of the function $\psi(\zeta)$ is determined by

$$Z = 2 + \frac{1}{\pi i} \int_1^{1/k} \frac{\eta^-(\xi)[\eta'(\xi)]^+ - \eta^+(\xi)[\eta'(\xi)]^-}{\eta^+(\xi)\eta^-(\xi)} d\xi, \quad (4.32)$$

where $\eta^\pm(\xi) = \eta(\xi \pm i0)$, $[\eta'(\xi)]^\pm = \eta'(\xi \pm i0)$. On substituting the limiting values $\eta^\pm(\xi)$ and $[\eta'(\xi)]^\pm$ into the last formula we simplify it to the form $Z = 2 - \alpha_0 \alpha_1 I_0 / \pi$, where we denoted

$$I_0 = 2 \int_1^{1/k} \frac{[(\xi^2 - \lambda)(2\xi^2 - 1 - 1/k^2) + 2|p_2(\xi)|]\xi d\xi}{[\alpha_1^2(\xi^2 - \lambda)^2 + \alpha_0^2|p_2(\xi)|]\sqrt{|p_2(\xi)|}}. \quad (4.33)$$

By denoting $\mu_\pm = (1/k^2 \pm 1)/2$ and making the substitutions first $\zeta^2 = \mu_- \cos \theta + \mu_+$ and then $w = e^{i\theta}$, we derive

$$I_0 = -2i \int_{|w|=1} \frac{\{[\mu_-(w^2 + 1) + 2\mu_+ w](\mu_+ - \lambda) + 2w(\lambda\mu_+ - 1/k^2)\}dw}{g_0(w)}, \quad (4.34)$$

where

$$g_0(w) = \mu_-^2(\alpha_1^2 - \alpha_0^2)(w^2 + 1)^2 + 4\alpha_1^2(\mu_+ - \lambda)\mu_- w(w^2 + 1) + 4[\alpha_1^2(\mu_+ - \lambda)^2 + \alpha_0^2\mu_-^2]w^2. \quad (4.35)$$

This integral is evaluated by the theory of residues. The four zeros of the function $g_0(w)$ can be easily determined; they are

$$w_{1,2} = \delta_\pm + \sqrt{\delta_\pm^2 - 1}, \quad w_{3,4} = \delta_\pm - \sqrt{\delta_\pm^2 - 1}, \quad (4.36)$$

and

$$\delta_\pm = \frac{\alpha_1^2(\lambda - \mu_+) \pm \alpha_0 \sqrt{\alpha_1^2(\mu_+ - \lambda)^2 - \mu_-^2(\alpha_1^2 - \alpha_0^2)}}{\mu_-(\alpha_1^2 - \alpha_0^2)}. \quad (4.37)$$

The final formula for the number of inadmissible poles of the function $\psi(\zeta)$ becomes

$$Z = 2 - \alpha_* \sum_{j=1, \dots, 4; |w_j| < 1} \frac{[\mu_-(w_j^2 + 1) + 2\mu_+ w_j](\mu_+ - \lambda) + 2w_j(\lambda\mu_+ - 1/k^2)}{g_j}. \quad (4.38)$$

where $\alpha_* = \alpha_0/\alpha_1$ and

$$g_j = \mu_-^2(1 - \alpha_*^2)w_j(w_j^2 + 1) + \mu_-(\mu_+ - \lambda)(3w_j^2 + 1) + 2w_j[(\mu_+ - \lambda)^2 + \alpha_*^2\mu_-^2], \quad (4.39)$$

It turns out that two and only two zeros out of the four zeros w_j ($j = 1, 2, 3, 4$) lie inside the unit disc $|w| < 1$. As in the case when ζ_∞ is a finite point in the segment $(-1, 1)$ if $\gamma > 1$, then $Z = 0$, and the function $\Psi(z)$ is analytic everywhere in the domain D^e . If $\gamma < 1$, then $Z = 4$, and the function $\Psi(z)$ has four simple poles in the domain D^e . In the limiting case $\gamma = 1$ the function $\Psi(z)$ has singularities in the boundary of the domain D^e . Sample contours of symmetric equal-strength holes are given in Figure 3 (a)-(c) in the case $\zeta = \infty$ and $\gamma > 1$. In Figure 3(d), the parameter $\gamma < 1$, the function $\Psi(z)$ has four inadmissible poles, and in addition, the contour intersect each other; the solution does not exist.

5 Three holes: ζ_∞ is a finite point

Any triply connected domain D^e can be considered as the image by a conformal map $z = \omega(\zeta)$ of a parametric ζ -plane cut along three segments in the real axis, $l_0 = [-1/k, -1]$, $l_1 = [k_1, k_2]$, and $l_2 = [1, 1/k]$, where $0 < k < 1$, $-1 < k_1 < k_2 < 1$. Given the domain D^e such a map is unique. The point $z = \infty$ is the image of a certain point $\zeta_\infty = \zeta'_\infty + i\zeta''_\infty$, and, in general, the parameters ζ'_∞ , ζ''_∞ , k , k_1 , and k_2 cannot be prescribed and should be recovered from the solution.

Let \mathcal{R} be the hyperelliptic surface of the algebraic function $u^2 = p_3(\zeta)$, where

$$p_3(\zeta) = (\zeta^2 - 1)(\zeta^2 - 1/k^2)(\zeta - k_1)(\zeta - k_2). \quad (5.1)$$

We fix the single branch $f(\zeta)$ of the function $p_3^{1/2}(\zeta)$ in $\mathbb{C} \setminus l$, $l = l_0 \cup l_1 \cup l_2$, by the condition $f(\zeta) \sim \zeta^3$, $\zeta \rightarrow \infty$. The branch is pure imaginary on the cut sides,

$$f(\zeta) = \pm(-1)^m i |p_3^{1/2}(\xi)|, \quad \zeta = \xi \pm i0, \quad \xi \in l_m, \quad m = 0, 1, 2,$$

$$f(\xi) = |p_3^{1/2}(\xi)|, \quad -1 < \xi < k_1, \quad f(\xi) = -|p_3^{1/2}(\xi)|, \quad k_2 < \xi < 1. \quad (5.2)$$

Since $\operatorname{Re} F_+(\zeta) = 0$ and $\operatorname{Im} F_-(\zeta) = 0$ on l , the functions $iF_+(\zeta)$ and $F_-(\zeta)$ can analytically and symmetrically be continued onto the whole Riemann surface. The new functions, $F_1(\zeta, u)$ and $F_2(\zeta, u)$, are rational on the surface and satisfy the symmetry condition

$$\overline{F_j(\zeta_*, u_*)} = F_j(\zeta, u), \quad (\zeta, u) \in \mathcal{R}, \quad (5.3)$$

where $(\zeta_*, u_*) = (\bar{\zeta}, -u(\bar{\zeta}))$ is the point symmetrical to the point (ζ, u) with respect to the line along which the two sheets \mathbb{C}_1 and \mathbb{C}_2 of the surface are connected. Note that if $(\zeta, u) \in \mathbb{C}_1$, then $(\zeta_*, u_*) \in \mathbb{C}_2$.

At the points $(\zeta_\infty, u_\infty) \in \mathbb{C}_1$ and $(\zeta_\infty^*, u_\infty^*) \in \mathbb{C}_2$ ($\zeta_\infty^* = \bar{\zeta}_\infty$), both of the functions $F_1(\zeta, u)$ and $F_2(\zeta, u)$ have order-2 poles. At the branch points of the surface, they have simple poles (in the sense of Riemann surfaces). At the two infinite points of the surface they have order-2 zeros. On the first sheet of the surface, the functions $F_1(\zeta, u) = F_+(\zeta)$ and $F_2(\zeta, u) = F_-(\zeta)$ have the form

$$F_+(\zeta) = \frac{R_+(\zeta)}{f(\zeta)}, \quad F_-(\zeta) = \frac{iR_-(\zeta)}{f(\zeta)}, \quad (5.4)$$

where

$$\begin{aligned} R_\pm(\zeta) = & A_1^\pm + A_2^\pm \zeta + A_3^\pm \frac{f(\zeta) + f(\zeta_\infty)}{\zeta - \zeta_\infty} - A_3^\pm \frac{f(\zeta) - f(\bar{\zeta}_\infty)}{\zeta - \bar{\zeta}_\infty} \\ & + (A_4^\pm + iA_5^\pm) \frac{f(\zeta) + f(\zeta_\infty) + f'(\zeta_\infty)(\zeta - \zeta_\infty)}{(\zeta - \zeta_\infty)^2} \\ & - (A_4^\pm - iA_5^\pm) \frac{f(\zeta) - f(\bar{\zeta}_\infty) - f'(\bar{\zeta}_\infty)(\zeta - \bar{\zeta}_\infty)}{(\zeta - \bar{\zeta}_\infty)^2}, \end{aligned} \quad (5.5)$$

where A_j^\pm ($j = 1, 2, \dots, 5$) are free real constants. Analysis of these functions in a neighborhood of the point ζ_∞ yields

$$F_\pm(\zeta) = i^{1/2 \mp 1/2} \left(\frac{2(A_4^\pm + iA_5^\pm)}{(\zeta - \zeta_\infty)^2} + \frac{2A_3^\pm}{\zeta - \zeta_\infty} \right) + O(1), \quad \zeta \rightarrow \zeta_\infty. \quad (5.6)$$

By virtue of the behavior of the functions $F_{\pm}(\zeta) = -c_{-1}(b \pm \bar{a})(\zeta - \zeta_{\infty})^{-2} + O(1)$, $\zeta \rightarrow \zeta_{\infty}$, required, we immediately find

$$\begin{aligned} A_3^{\pm} &= 0, \quad A_4^+ = -\frac{1}{2}(c'\alpha^+ - c''\beta^+), \quad A_4^- = -\frac{1}{2}(c'\beta^- + c''\alpha^-), \\ A_5^+ &= -\frac{1}{2}(c'\beta^+ + c''\alpha^+), \quad A_5^- = \frac{1}{2}(c'\alpha^- - c''\beta^-). \end{aligned} \quad (5.7)$$

The coefficients A_1^{\pm} and A_2^{\pm} are still not determined in the expression of the function $\omega'(\zeta)$ that is

$$\begin{aligned} \omega'(\zeta) &= \frac{1}{2\bar{a}f(\zeta)} \left[A_1 + A_2\zeta + (A_4 + iA_5) \frac{f(\zeta) + f(\zeta_{\infty}) + f'(\zeta_{\infty})(\zeta - \zeta_{\infty})}{(\zeta - \zeta_{\infty})^2} \right. \\ &\quad \left. - (A_4 - iA_5) \frac{f(\zeta) - f(\bar{\zeta}_{\infty}) - f'(\bar{\zeta}_{\infty})(\zeta - \bar{\zeta}_{\infty})}{(\zeta - \bar{\zeta}_{\infty})^2} \right], \end{aligned} \quad (5.8)$$

where $A_j = A_j^+ - iA_j^-$. To determine the coefficients A_1 and A_2 , we need to guarantee that the function $z = \omega(\zeta)$ is a single-valued map that is to force the function $\omega'(\zeta)$ to meet the three conditions

$$\int_{l_m} \omega'(\zeta) d\zeta = 0, \quad m = 0, 1, 2. \quad (5.9)$$

or, equivalently,

$$I_{m0}A_1 + I_{m1}A_2 = -J_m, \quad m = 0, 1, 2, \quad (5.10)$$

where

$$\begin{aligned} I_{mj} &= \int_{l_m} \frac{\xi^j d\xi}{f(\xi)}, \quad j = 0, 1, \\ J_m &= (A_4 + iA_5) \int_{l_m} \frac{[f(\zeta_{\infty}) + f'(\zeta_{\infty})(\xi - \zeta_{\infty})]d\xi}{f(\xi)(\xi - \zeta_{\infty})^2} \\ &\quad + (A_4 - iA_5) \int_{l_m} \frac{[f(\bar{\zeta}_{\infty}) + f'(\bar{\zeta}_{\infty})(\xi - \bar{\zeta}_{\infty})]d\xi}{f(\xi)(\xi - \bar{\zeta}_{\infty})^2}, \quad m = 0, 1, 2. \end{aligned} \quad (5.11)$$

The first two equations in (5.10) constitute an inhomogeneous system of two complex equations with respect to complex constants A_1 and A_2 . The coefficients of the system, the integrals I_{mj} ($j, m = 0, 1$) are the A -periods of the abelian integrals

$$\int_{(1/k, 0)}^{(\zeta, u(\zeta))} \frac{\xi^j d\xi}{u(\xi)}, \quad j = 0, 1, \quad (5.12)$$

associated with the genus-2 Riemann surface \mathcal{R} of the algebraic function $u^2(\xi) = p_3(\xi)$. Therefore the 2×2 matrix $\{I_{mj}\}$ ($j, m = 0, 1$) is not singular, and the unique solution is given by

$$A_1 = \frac{J_1 I_{01} - J_0 I_{11}}{\Delta}, \quad A_2 = \frac{J_0 I_{10} - J_1 I_{00}}{\Delta}, \quad (5.13)$$

where $\Delta = I_{00}I_{11} - I_{01}I_{10}$. The third equation in (5.10) is transformed to the form

$$J_1(I_{01}I_{20} - I_{00}I_{21}) + J_0(I_{10}I_{21} - I_{11}I_{20}) + J_2\Delta = 0 \quad (5.14)$$

and satisfied identically. This is due to the fact that the corresponding abelian integrals in the right hand-side in (5.10) can be represented as a linear combination of the two basis integrals (5.12).

Since the parameters k , k_1 , k_2 , and $\zeta_\infty = \zeta'_\infty + i\zeta''_\infty$ are free, the derivative $\omega'(\zeta)$ generates a five-parametric family of conformal mappings (we do not count the two free scaling parameters $c_{-1} = c' + ic''$) which transforms the slit domain $\mathbb{C} \setminus l$ into the triple connected domain D^e . By integrating the function (5.8) we find the integral representation of the conformal map

$$\omega(\zeta) = \frac{1}{2\bar{a}} \left\{ -\frac{A_4 + iA_5}{\zeta - \zeta_\infty} + \frac{A_4 - iA_5}{\zeta - \bar{\zeta}_\infty} + \int_{\zeta_0}^{\zeta} [A_1 + A_2\xi + (A_4 + iA_5) \right. \\ \left. \times \frac{f(\zeta_\infty) + f'(\zeta_\infty)(\xi - \zeta_\infty)}{(\xi - \zeta_\infty)^2} + (A_4 - iA_5) \frac{f(\bar{\zeta}_\infty) + f'(\bar{\zeta}_\infty)(\xi - \bar{\zeta}_\infty)}{(\xi - \bar{\zeta}_\infty)^2} \right] \frac{d\xi}{f(\xi)} \Big\}. \quad (5.15)$$

To find the actual profile of the holes L_m , we let ζ run the cuts l_m and obtain

$$z = \mathcal{I}(\zeta) \pm i\mathcal{J}(-1/k, \zeta), \quad \zeta \in l_0^\mp, \quad z \in L_0, \\ z = \mathcal{I}(\zeta) + \mathcal{J}(-1, k_1) \mp i\mathcal{J}(k_1, \zeta), \quad \zeta \in l_1^\mp, \quad z \in L_1, \\ z = \mathcal{I}(\zeta) + \mathcal{J}(-1, k_1) - \mathcal{J}(k_2, 1) \pm i\mathcal{J}(1, \zeta), \quad \zeta \in l_2^\mp, \quad z \in L_2, \quad (5.16)$$

where we denoted

$$\mathcal{I}(\zeta) = \frac{1}{2\bar{a}} \left(-\frac{A_4 + iA_5}{\zeta - \zeta_\infty} + \frac{A_4 - iA_5}{\zeta - \bar{\zeta}_\infty} \right), \\ \mathcal{J}(d, \zeta) = \frac{1}{2\bar{a}} \int_d^\zeta \left[A_1 + A_2\xi + (A_4 + iA_5) \frac{f(\zeta_\infty) + f'(\zeta_\infty)(\xi - \zeta_\infty)}{(\xi - \zeta_\infty)^2} \right. \\ \left. + (A_4 - iA_5) \frac{f(\bar{\zeta}_\infty) + f'(\bar{\zeta}_\infty)(\xi - \bar{\zeta}_\infty)}{(\xi - \bar{\zeta}_\infty)^2} \right] \frac{d\xi}{|f(\xi)|} \quad (5.17)$$

The function $\omega'(\zeta)$ may not have zeros in the slit domain $\mathbb{C} \setminus l$. Otherwise the functions $\psi(\zeta)$ has unacceptable poles. As in the case of doubly connected domain we introduce a function $\eta(\zeta)$ which share the zeros with the function $\omega'(\zeta)$ and is free of singularities of $\omega'(\zeta)$,

$$\omega'(\zeta) = \frac{\eta(\zeta)}{2\bar{a}f(\zeta)(\zeta - \zeta_\infty)^2(\zeta - \bar{\zeta}_\infty)^2}, \quad (5.18)$$

where

$$\eta(\zeta) = (A_1 + A_2\zeta)(\zeta - \zeta_\infty)^2(\zeta - \bar{\zeta}_\infty)^2 + (A_4 + iA_5)(\zeta - \bar{\zeta}_\infty)^2[f(\zeta) + f(\zeta_\infty) \\ + f'(\zeta_\infty)(\zeta - \zeta_\infty)] - (A_4 - iA_5)(\zeta - \zeta_\infty)^2[f(\zeta) - f(\bar{\zeta}_\infty) - f'(\bar{\zeta}_\infty)(\zeta - \bar{\zeta}_\infty)]. \quad (5.19)$$

The zero counting formula applied yields that the number of zeros, Z , of the function $\eta(\zeta)$ in the slit domain is

$$Z = 5 + \frac{1}{2\pi i} \sum_{m=0}^2 \int_{l_m} \frac{\eta'(\zeta)d\zeta}{\eta(\zeta)}. \quad (5.20)$$

Here, we used the asymptotics at infinity

$$f(\zeta) \sim \zeta^3, \quad f'(\zeta) \sim 3\zeta^2, \\ \eta(\zeta) \sim (A_2 + 2iA_5)\zeta^5, \quad \eta'(\zeta) \sim 5(A_2 + 2iA_5)\zeta^4, \quad \zeta \rightarrow \infty, \quad (5.21)$$

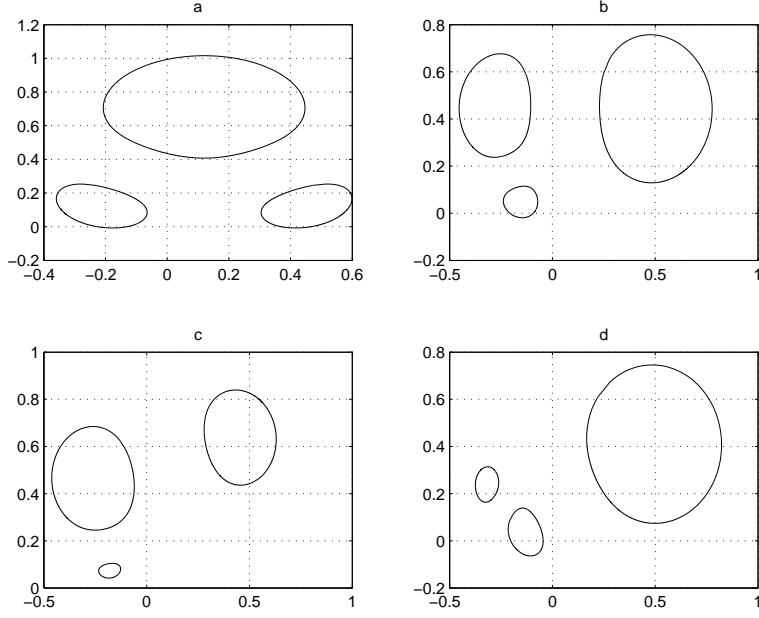


Figure 4: The contours L_0 , L_1 , and L_2 when ζ_∞ is a finite point, $\sigma_1^\infty = \sigma_2^\infty = 1$, $\tau^\infty = \tau = 0$, $p = 5$, $(\gamma = \infty)$, $c' = 1, c'' = 0$, and $k_1 = -0.8$. (a): $\zeta_\infty = i$, $k = 0.2$, $k_2 = 0.8$, (b): $\zeta_\infty = 1 + i$, $k = 0.2$, $k_2 = 0.8$, (c): $\zeta_\infty = 1 + i$, $k = 0.5$, $k_2 = 0.8$, (d): $\zeta_\infty = 1 + i$, $k = 0.1$, $k_2 = 0$. In all cases (a)-(d), the function $\Psi(z)$ has two poles in the domain D^e .

and the limit as $R \rightarrow \infty$ of the integral over a circle Γ_R of radius R centered at the origin

$$\lim_{R \rightarrow \infty} \int_R \frac{\eta'(\zeta) d\zeta}{\eta(\zeta)} = 5. \quad (5.22)$$

On computing the integrals in (5.20) it is possible to establish that if $\gamma = |a/b| > 1$, then $Z = 2$, and when $\gamma < 1$, the function $\omega'(\zeta)$ has even more zeros, $Z = 8$. The limiting case $\gamma = 1$ is singular as in the cases of simply and doubly connected domains. Thus, the conformal map which transforms any finite point ζ_∞ of the slit domain into the infinite point $z = \infty$ gives rise to a finite number of poles or singularities on the holes boundaries for the function $\Psi(z)$. This means that such a family of maps cannot be employed to identify equally strong holes. At the same time, when $\gamma > 1$, the map $z = f(\zeta)$ given by (5.16) generates some contours L_m which do not intersect each other. Samples of such contours are given in Figure 4.

6 n holes: $n \geq 3$ and $\zeta_\infty = \infty$

The family of mappings derived in the previous section gives rise to unacceptable poles of the complex potential $\Psi(z)$. All the mappings share the same property, the infinite point $z = \infty$ is the image of a certain finite point ζ_∞ in the slit domain. Since the case $\omega(\infty) = \infty$ cannot be extracted from the solution derived in Section 5, we consider this case separately. Also, for generality, we assume that n is not just equal to 3, but any finite integer $n \geq 3$. We confine ourselves to the family of domains D^e which are the images of slit domains \mathcal{D}^e such that all the n slits lie in the same line, $\mathcal{D}^e = \mathbb{C} \setminus l$, $l = l_0 \cup \dots \cup l_{n-1}$, and $l_j = [k_{2j}, k_{2j+1}]$, $j = 0, \dots, n-1$, $k_{2n-1} = -k_0 = 1$, $-1 < k_1 < \dots < k_{2n-2} < 1$. The

function $\omega(\zeta)$ has a simple pole in the vicinity of the infinite point and for large z can be represented by (3.1). We emphasize that not every triply connected domain D^e is the image of a slit domain \mathcal{D}^e such that $z = \infty$ is the image of $\zeta = \infty$. Needless to say that not every n -connected ($n \geq 4$) domain D^e is the image of the exterior of n slits lying in the same line. On studying the family of mappings $\omega : \mathcal{D}^e \rightarrow D^e$ when $\omega(\infty) = \infty$ we try to find a set of equal-strength holes such that $\omega'(\zeta)$ does not have zeros in \mathcal{D}^e , and therefore the function $\Psi(z)$ is analytic everywhere in the domain D^e .

First we fix the branch $f(\zeta)$ of the function $p_n^{1/2}(\zeta)$,

$$p_n(\zeta) = \prod_{j=0}^{2n-1} (\zeta - k_j), \quad (6.1)$$

in the domain \mathcal{D}^e by the condition $f(\zeta) \sim \zeta^n$. The functions $F_{\pm}(\zeta)$ are bounded at infinity, and their counterparts defined in the Riemann surface have simple poles at the branch points of the surface \mathcal{R} . We have

$$F_+(\zeta) = \frac{1}{f(\zeta)} \sum_{j=1}^{n+1} A_j^+ \zeta^{j-1} + iA_0^+, \quad F_-(\zeta) = \frac{i}{f(\zeta)} \sum_{j=1}^{n+1} A_j^- \zeta^{j-1} + A_0^-, \quad (6.2)$$

where A_j^{\pm} are arbitrary real constants. By expanding these functions for large z and comparing these expansions with the asymptotics of $F_{\pm}(\zeta)$ in (3.3) we find

$$A_n^{\pm} = -kA_{n+1}^{\pm}, \quad A_{n+1}^- = \alpha^- c'' + \beta^- c', \quad A_{n+1}^+ = \alpha^+ c' - \beta^+ c'', \\ A_0^- = \alpha^- c' - \beta^- c'', \quad A_0^+ = \alpha^+ c'' + \beta^+ c', \quad (6.3)$$

where $k = \frac{1}{2}(k_1 + k_2 + \dots + k_{2n-2})$. The derivative of the conformal map $\omega'(\zeta)$

$$\omega'(\zeta) = \frac{1}{2\bar{a}} \left[iA_0^+ - A_0^- + \frac{1}{f(\zeta)} \sum_{j=1}^{n+1} (A_j^+ - iA_j^-) \zeta^{j-1} \right] \quad (6.4)$$

has to generate a one-to-one map. This is guaranteed by the following n complex conditions:

$$\int_{l_m} \omega'(\zeta) d\zeta = 0, \quad m = 0, 1, \dots, n-1. \quad (6.5)$$

These conditions can be rewritten as

$$\sum_{j=0}^n a_{mj} (A_{j+1}^+ - iA_{j+1}^-) = 0, \quad m = 0, 1, \dots, n-1. \quad (6.6)$$

Here,

$$a_{mj} = \int_{l_m} \frac{\zeta^j d\zeta}{f(\zeta)}, \quad m = 0, 1, \dots, n-1, \quad j = 0, 1, \dots, n. \quad (6.7)$$

The integrals a_{mj} ($m, j = 0, 1, \dots, n-2$) are the A -periods of the abelian integrals

$$\int_{(1,0)}^{(\zeta, u(\zeta))} \frac{\xi^j d\xi}{u(\xi)}, \quad j = 0, 1, \dots, n-2, \quad (6.8)$$

associated with the genus- $(n-1)$ Riemann surface \mathcal{R} of the algebraic function $u^2(\xi) = p_n(\xi)$. Therefore the matrix a_{mj} ($m, j = 0, 1, \dots, n-2$) is not singular. Denote

$$I_{mj} = \int_{l_m^+} \frac{\xi^j d\xi}{|f(\xi)|}, \quad m = 0, 1, \dots, n-1, \quad j = 0, 1, \dots, n. \quad (6.9)$$

The coefficients A_j^\pm are uniquely determined through the known coefficients A_{n+1}^\pm from the nonsingular system

$$\sum_{j=0}^{n-2} I_{mj} A_{j+1}^\pm = -(I_{mn} - kI_{mn-1}) A_{n+1}^\pm, \quad m = 0, 1, \dots, n-2. \quad (6.10)$$

The last equation in (6.6) is satisfied automatically because the basis of the abelian integrals (6.8) has dimension $n-1$, and the $n \times n$ matrix

$$\begin{pmatrix} I_{00} & \dots & I_{0n-2} & I_{0n} - kI_{0n-1} \\ \dots & \dots & \dots & \dots \\ I_{n-10} & \dots & I_{n-1n-2} & I_{n-1n} - kI_{n-1n-1} \end{pmatrix} \quad (6.11)$$

is singular.

To determine the number of zeros of the function $\omega'(\zeta)$, we introduce the function

$$\eta(\zeta) = \sum_{j=1}^{n+1} (A_j^+ - iA_j^-) \zeta^{j-1} + (iA_0^+ - A_0^-) f(\zeta). \quad (6.12)$$

The functions $\eta(\zeta)$ and $\omega'(\zeta)$ share their zeros. The number of zeros of the function $\eta(\zeta)$ coincides with the number of inadmissible poles of the function $\psi(\zeta)$ and is given by

$$Z = \frac{1}{2\pi i} \left(\sum_{j=0}^{n-1} \int_{l_j} + \lim_{R \rightarrow \infty} \int_{\Gamma_R} \right) \frac{\eta'(\zeta) d\zeta}{\eta(\zeta)} = n + \frac{1}{2\pi i} \sum_{j=0}^{n-1} \int_{l_j} \frac{\eta'(\zeta) d\zeta}{\eta(\zeta)}. \quad (6.13)$$

Our numerical tests implemented for the case $n=3$ reveal that if $\gamma = |a/b| > 1$, then $Z=0$, and the function $\Psi(z)$ is analytic in D^e and continuous up to the boundary. If $\gamma < 1$, then $Z=6$, and the function $\Psi(z)$ has six poles in the domain D^e . When $|a| = |b|$, Z is not an integer number, and the function $\Psi(z)$ has singularities in the boundary of D . As in the simply and doubly connected cases, if $|a| \leq |b|$, then the problem does not have solutions. When $|a| > |b|$, the solution exists, and the conformal map $z = \omega(\zeta)$ is defined up to seven arbitrary constants, an additive constant, two scaling parameters c' and c'' , and the four parameters k_j ($j=1, \dots, 4$). Integrating (6.4) and employing (6.5), we have

$$\begin{aligned} z &= \frac{1}{2a} [(iA_0^+ - A_0^-) \zeta \mp iJ(-1, \zeta)] + B, \quad \zeta \in l_0^\pm, \quad z \in L_0, \\ z &= \frac{1}{2a} [(iA_0^+ - A_0^-) \zeta + J(k_1, k_2) \pm iJ(k_2, \zeta)] + B, \quad \zeta \in l_1^\pm, \quad z \in L_1, \\ z &= \frac{1}{2a} [(iA_0^+ - A_0^-) \zeta + J(k_1, k_2) - J(k_3, k_4) \mp iJ(k_4, \zeta)] + B, \quad \zeta \in l_2^\pm, \quad z \in L_2. \end{aligned} \quad (6.14)$$

Here, B is an additive constant and without loss can be taken zero, and J is the real integral

$$J(\alpha, \beta) = \int_\alpha^\beta \frac{1}{|f(\xi)|} \sum_{j=1}^4 (A_j^+ - iA_j^-) \xi^{j-1} d\xi. \quad (6.15)$$

Thus, if $|a/b| > 1$, the solution of the problem exists and possesses six real parameters (two of them are the scaling parameters c' , and c'').

Figures 5(a)-(c) and 6(a)-(c) show how the change of the loading parameter γ and the conformal mapping parameters affects the profiles of equal-strength cavities in the

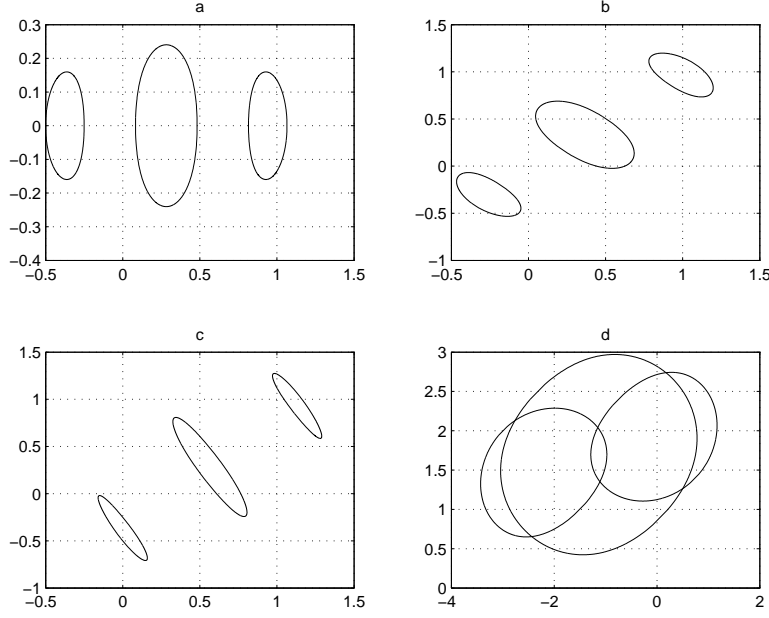


Figure 5: Three equal-strength holes when $\zeta_\infty = \infty$, $k_1 = -k_4 = -0.5$, $k_2 = -k_3 = -0.4$, $c' = 1$, $\sigma_1^\infty = \sigma_2^\infty = 1$. (a): $\tau^\infty = \tau = 0$, $p = 5$ ($\gamma = \infty$), $c'' = 0$. (b): $\tau^\infty = \tau = 1$, $p = 5$ ($\gamma = 4.1231$), $c'' = 1$. (c): $\tau^\infty = \tau = 1$, $p = 2$ ($\gamma = 1.4142$), $c'' = 1$. (d): $\tau^\infty = -5$, $\tau = p = 1$ ($\gamma = 0.2$), $c'' = 1$. In the case (d), the function $\Psi(z)$ has six inadmissible poles.

case $n = 3$ and when $\zeta_\infty = \infty$. Figures 5(d) and 6(d) give samples of the contours L_j when $\gamma < 1$. In both cases the function $\Psi(z)$ has six poles in the domain D^e . While in Figure 6(d) the contours L_j do not have common points, in the case presented in Figure 5(d) all three contours intersect with each other.

Conclusions. We have analyzed the inverse plane problem of constructing n equal-strength cavities in an unbounded elastic body when constant loading is applied at infinity and to the cavities boundary. By advancing the method of conformal mappings employed in [4] for $n = 1$ and $n = 2$ (two symmetric holes) to general doubly and triply connected domains we have found a four- and seven-parametric family of mappings and therefore a four- and seven-parametric family of two and three equal-strength cavities, respectively. In both cases two out of four and seven free parameters, respectively, are scaling parameters. For the doubly connected problem, the map ω transforms a slit domain \mathcal{D}^e , the exterior of two slits $[-1/k, -1]$ and $[1, 1/k]$, into the elastic domain D^e , the exterior of the holes, and $\omega(\zeta_\infty) = \infty$, $\zeta_\infty \in (-1, 1)$. For the triply connected problem, we analyzed two cases of the preimage of the infinite point, ζ_∞ is a finite point and $\zeta_\infty = \infty$. In the former case, \mathcal{D}^e is the exterior of three slits $[-1/k, -1]$, $[k_1, k_2]$, and $[1, 1/k]$, while in the second case the slits are $[-1, k_1]$, $[k_2, k_3]$, and $[k_4, 1]$. The conformal mappings are derived in terms of elliptic integrals for $n = 2$ and hyperelliptic integrals for $n = 3$. We have also analyzed the zeros of the derivative $\omega'(\zeta)$ of the conformal mapping and shown that these zeros, if exist, generate inadmissible poles of the solution. If $\gamma = |a/b| > 1$ (a and b are complex loading parameters), then the Kolosov-Muskhelishvili potential $\Psi(z)$ is free of poles when $n = 1$, $n = 2$, and $n = 3$. In the triply connected case the conformal map is chosen such that $\omega(\infty) = \infty$. Otherwise,

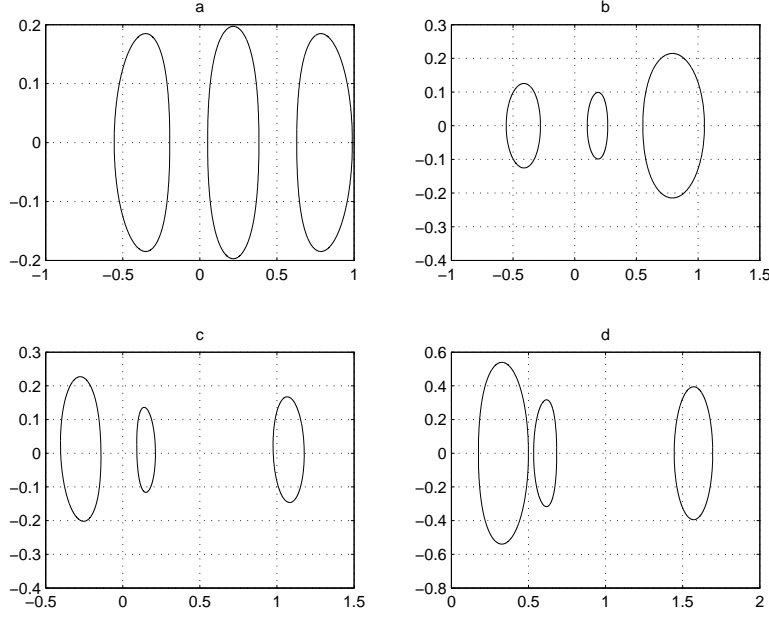


Figure 6: Three equal-strength holes when $\zeta_\infty = \infty$, $c' = 1$, $c'' = 0$. (a): $\sigma_1^\infty = 0$, $\sigma_2^\infty = 1$, $\tau^\infty = \tau = 0$, $p = 5$ ($\gamma = 9$), $k_1 = -k_4 = -0.35$, $k_2 = -k_3 = -0.3$. (b): $k_1 = -0.5$, $k_2 = -0.3$, $k_3 = 0$, $k_4 = 0.1$, and the other parameters are the same as in (a). (c): $\sigma_1^\infty = \sigma_2^\infty = 0$, $\tau^\infty = 1$, $\tau = 5$, $p = 1$ ($\gamma = 5.0990$), $k_1 = -0.35$, $k_2 = -0.3$, $k_3 = 0$, $k_4 = 0.5$. (d): $\sigma_1^\infty = \sigma_2^\infty = \tau^\infty = 1$, $\tau = 0.5$, $p = 1$ ($\gamma = 0.5$), and the other parameters are the same as in (c). In case (d), the function $\Psi(z)$ has six inadmissible poles.

if $\gamma < 1$, the function $\Psi(\zeta)$ has two, four, and six inadmissible poles, respectively. If $\gamma = 1$, then the singularities lie in the contours of the holes. It has been discovered that when $\gamma < 1$ and small enough, the contours intersect each other. If $n = 3$ and ζ_∞ is a finite point, then $\Psi(\zeta)$ has two and eight poles for the cases $\gamma > 1$ and $\gamma < 1$, respectively. We have also derived an integral representation in terms of hyperelliptic integrals for a $2n$ -parametric family (two of them are scaling parameters) of conformal mappings for the case $n \geq 4$ which assume that $\omega(\infty) = \infty$, and the slits lie in the same line. We conjecture that the function $\Psi(z)$ is free of poles and the solution exists when $\gamma > 1$ and the function $\Psi(z)$ has $2n$ poles in the domain D^e when $\gamma < 1$.

References

- [1] Y.A. ANTIPOV AND V.V. SILVESTROV, *Method of Riemann surfaces in the study of supercavitating flow around two hydrofoils in a channel*, Physica D, 235 (2007), pp. 72-81.
- [2] N.V. BANICHUK, *Problem of optimization of the shape of a hole in a plate under bending*, Solid Mechanics – Izv. AN SSSR, Mekhanika Tverdogo Tela, no.3 (1977), pp. 81-88.
- [3] G.P. CHEREPANOV, *Inverse problem of the theory of elasticity*, Solid Mechanics – Izv. AN SSSR, Mekhanika Tverdogo Tela, no.3 (1966), 81-82.

- [4] G.P. CHEREPANOV, *Inverse problems of the plane theory of elasticity*, J. Appl. Math. Mech., 38 (1974), pp. 915-931.
- [5] R. COURANT, *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces*, Interscience Publishers, Inc., New York, N.Y., 1950.
- [6] J.D. ESHELBY, *The determination of the elastic field of an ellipsoidal inclusion, and related problems*, Proc. Roy. soc. London A, 241 (1957), pp. 376-396.
- [7] Y. GRABOVSKY AND R.V. KOHN, *Microstructures minimizing the energy of a two phase elastic composite in two space dimensions. II.: The Vigdergauz microstructure*, J. Mech. Phys. Solids, 43, pp. 949- 972.
- [8] H. KANG, E. KIM AND G. W. MILTON, *Inclusion pairs satisfying Eshelby's uniformity property*, SIAM, J. Appl. Math. 69 (2008), pp. 577-595.
- [9] M.V. KELDYSH, *Conformal mappings of multiply connected domains on canonical domains*, Uspekhi Matem. Nauk 6 (1939), pp. 90-119.
- [10] A.S. KOSMODAMIANSKII, *Stress state of anisotropic media with holes and cavities*, Viscza shkola, Kiev, 1976.
- [11] N.I. MUSKHELISHVILI, *Some Basic Problems of the Mathematical Theory of Elasticity*, P. Noordhoff, Ltd., Groningen, 1963.
- [12] H. NEUBER, *Zur Optimierung der Spannungskonzentration*, in Continuum Mechanics and Related Problems of Analysis, Nauka, Moscow (1972), pp. 375-380.
- [13] C.-Q. RU AND P. SCHIAVONE, *On the elliptic inclusion in anti-plane shear*, Mech. Math. Solids, 1 (1996), pp. 327-333.
- [14] G.N. SAVIN, *Stress Distribution Around Holes*, Naukova Dumka, Kiev, 1968 (NASA Tech. Trans., Washington D.C., 1970).
- [15] G.P. SENDECKYJ, *Elastic inclusion problems in plane elastostatics*, Int. J. Solids Structures 6 (1970), pp. 1535-1543.
- [16] R.-J. SHIH AND L.T. WHEELER, *Two-dimensional inhomogeneities of minimum stress concentration*, Quart. Appl. Math. 64 (1986), pp. 567-582.
- [17] G. SPRINGER, *Introduction to Riemann Surfaces*, Addison-Wesley, Reading, MA, 1956.
- [18] S.B. VIGDERGAUZ, *Integral equation of the inverse problem of the plane theory of elasticity*, J. Appl. Math. Mech., 40 (1976), pp. 518-522.
- [19] S.B. VIGDERGAUZ, *Effective elastic parameters of a plate with a regular system of equal-strength holes*, Solid Mechanics – Izv. AN SSSR, Mekhanika Tverdogo Tela, 21, no.2 (1986), pp.165-169.